A quantitative assessment of the Solow model

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Growth Theory

Introduction

The Solow model makes predictions about:

- Steady state differences in income per capita across countries.
- The behavior of steady state growth.
- The phenomena of growth miracles/disasters.
- Convergence of income per capita across countries.

We can test these predictions.

Measurement

I am going to use data from the Penn World Tables.

To measure $1 - \alpha$, I use the labor share for each country.

To measure the labor input, I multiply the number of working people by the average hour worked. This allows countries to differ in the numbers of hours worked by person which I take as exogenous.

Steady state differences in income per capita

Steady state differences in income per capita

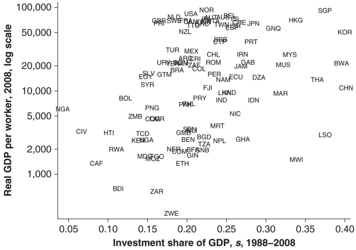
The Solow model makes some key predictions about the level of output per worker across countries:

$$\left(\frac{Y(t)}{L(t)}\right)^{*} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \exp(\psi\, u) A(t) \tag{1}$$

- Output per capita is increasing in the savings rate: $s = \frac{I}{Y}$.
- Output per capita is decreasing in the population growth rate.
- Output per capita is increasing in education levels.
- Output per capita is increasing in productivity (TFP).

GDP per capita and the savings rate

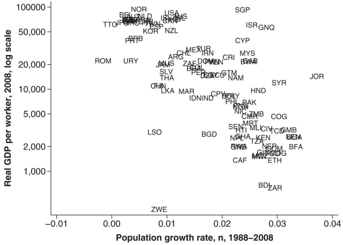
FIGURE 2.6 **GDP PER WORKER VERSUS THE INVESTMENT RATE**



Indeed, we find a positive correlation between the investment rate and GDP per capita.

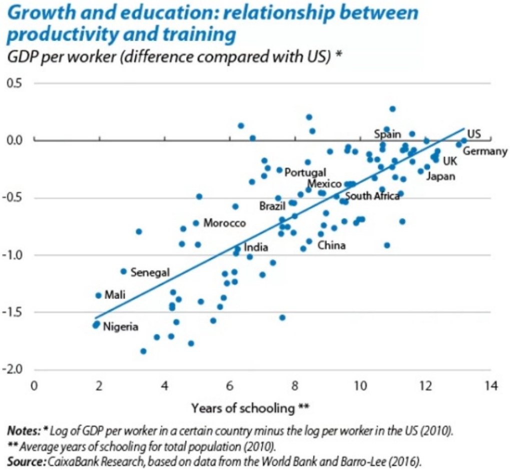
GDP per capita and the population growth rate

FIGURE 2.7 GDP PER WORKER VERSUS POPULATION GROWTH RATES



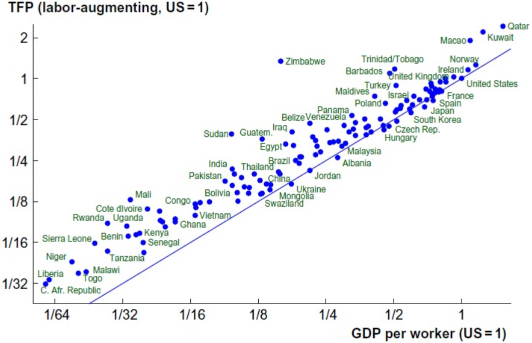
Indeed, we find a negative correlation between the population growth rate and GDP per capita.

GDP per capita and education



Indeed, we find a positive correlation between educaton and GDP per capita.

GDP per capita and productivity



Indeed, we find a positive correlation between productivity and GDP per capita.

A quantitative assessment

Instead of just looking qualitatively at the data, Mankiw et al. (1992) evaluate the quantitative performance of the model. Assuming that all countries are in steady state, they start from:

$$\left(\frac{Y(t)}{L(t)}\right)^{*} = A(t)\exp(\psi\, u)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{2}$$

$$y(t)^{*} = A(0)\exp(gt)\exp(\psi\, u)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{3}$$

$$\ln y(t)^{*} = \ln A(0) + gt + \frac{\alpha}{1-\alpha}\ln s - \frac{\alpha}{1-\alpha}\ln(n+g+\delta) + \psi\, u \tag{4}$$

Assuming that $\ln A(0) + gt = \beta_0 + \epsilon$, i.e., the level of technology is random across countries, and $g + \delta = 0.05$ this can be estimated by linear OLS:

$$\ln y(t) = \beta_0 + \beta_1 \ln s + \beta_2 \ln(n + 0.05) + \beta_3 u + \epsilon(t). \tag{5}$$

A quantitative assessment II

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	6.89	7.81	8.63
	(1.17)	(1.19)	(2.19)
ln(I/GDP)	0.69	0.70	0.28
	(0.13)	(0.15)	(0.39)
$\ln(n + g + \delta)$	−1.73	−1.50	−1.07
	(0.41)	(0.40)	(0.75)
ln(SCHOOL)	0.66	0.73	0.76
	(0.07)	(0.10)	(0.29)
$\overline{R}^2$	0.78	0.77	0.24

- The three variables together explain almost 80% in cross-country variation in GDP per capita.
- All signs are the expected sign.
- The implied α is reasonable.
- However, the data rejects $\beta_1 = -\beta_2$.

Drawbacks of the linear regression model

The linear regression model approach has several drawbacks:

- Endogeneity of variables is a serious issue. Unobservables, such as management capacity, are likely correlated with education (and savings rates).
- We have to assume a steady state.
- The regression model chooses coefficients that best fit the data, but they may be unreasonable economically.

Development accounting

Development accounting is an alternative approach to ask how good the Solow model is in explaining cross country income differences. It relates observable inputs to GDP per capita through the production function:

$$Y(t) = K(t)^{\alpha} (A(t)H(t))^{1-\alpha} \tag{6}$$

$$Y(t)^{1-\alpha} = \left(\frac{K(t)}{Y(t)}\right)^{\alpha} (A(t)H(t))^{1-\alpha} \tag{7}$$

$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)H(t) \tag{8}$$

$$\frac{Y(t)}{L(t)} = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi u). \tag{9}$$

Note, any cross country differences in s or n, the heart of the Solow model, should be reflected in the capital-to-output ratio.

Development accounting II

$$y(t) = \frac{Y(t)}{L(t)} = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t) \exp(\psi u). \tag{10}$$

- We are going to assume $\alpha = 0.3$.
- We use micro estimates of the return to schooling for ψ.
- Note, different from the linear regression framework, we now fix values for α and ψ instead of letting the regression choose those that best fit the data. Moreover, we do not impose that all economies are in steady state, i.e., the production function holds in and out of steady state.

Development accounting III

Taking the U.S. as reference, we can ask what factor explains differences in output per capita relative to the U.S.:

$$\frac{y(t)}{y^{US}(t)} = \frac{\left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{K^{US}(t)}{Y^{US}(t)}\right)^{\frac{\alpha}{1-\alpha}}} \frac{A(t)}{A^{US}(t)} \exp(\psi(u - u^{US})). \tag{11}$$

For example, the U.S. has around 11 years of schooling while the poorest countries have only 3. With a 10% return on schooling, we have:

$$\exp(0.1(3 - 11)) = 0.45, \tag{12}$$

i.e., differences in education can explain a 55% lower output per capita in the poorest countries.

Development accounting IV

Can capital intensity explain output per worker differences?

$$\frac{y(t)}{y^{US}(t)} = \frac{\left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{K^{US}(t)}{Y^{US}(t)}\right)^{\frac{\alpha}{1-\alpha}}} \tag{13}$$

$$\frac{y(t)}{y^{US}(t)} = \frac{\left(\frac{k(t)}{y(t)}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{k^{US}(t)}{y^{US}(t)}\right)^{\frac{\alpha}{1-\alpha}}} \tag{14}$$

$$\frac{y(t)}{y^{US}(t)} = \left(\frac{k(t)}{k^{US}(t)}\right)^{\alpha}. \tag{15}$$

To explain income per worker differences by a factor of 10 (U.S. relative to India in 2015), we need differences in capital per worker by a factor of 1000! The reason are diminishing marginal returns to capital. The actual difference was 9!

Development accounting V

A yet different way to see the same point is to rewrite the decomposition in terms of marginal products:

$$\frac{y(t)}{y^{US}(t)} = \frac{\left(\frac{\alpha}{MPK(t)}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{\alpha}{MPK^{US}(t)}\right)^{\frac{\alpha}{1-\alpha}}} \tag{16}$$

$$\frac{y(t)}{y^{US}(t)} = \left(\frac{MPK^{US}(t)}{MPK(t)}\right)^{\frac{1-\alpha}{\alpha}} . \tag{17}$$

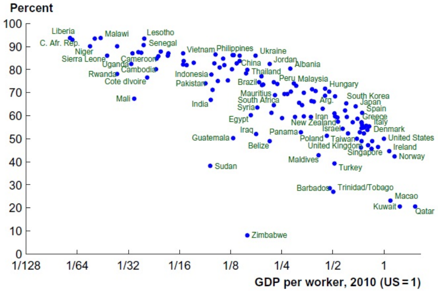
To explain that the U.S. is ten times richer than India in 2015, we require that the marginal product of capital is 100 times higher in India. With such huge returns on capital in India, world investors would certainly start investing in India.

Development accounting VI

	GDP per worker, y	Capital/GDP $(K/Y)^{\alpha/(1-\alpha)}$	Human capital, h	TFP	Share due to TFP
United States	1.000	1.000	1.000	1.000	–
Hong Kong	0.854	1.086	0.833	0.944	48.9%
Singapore	0.845	1.105	0.764	1.001	45.8%
France	0.790	1.184	0.840	0.795	55.6%
Germany	0.740	1.078	0.918	0.748	57.0%
United Kingdom	0.733	1.015	0.780	0.925	46.1%
Japan	0.683	1.218	0.903	0.620	63.9%
South Korea	0.598	1.146	0.925	0.564	65.3%
Argentina	0.376	1.109	0.779	0.435	66.5%
Mexico	0.338	0.931	0.760	0.477	59.7%
Botswana	0.236	1.034	0.786	0.291	73.7%
South Africa	0.225	0.877	0.731	0.351	64.6%
Brazil	0.183	1.084	0.676	0.250	74.5%
Thailand	0.154	1.125	0.667	0.206	78.5%
China	0.136	1.137	0.713	0.168	82.9%
Indonesia	0.096	1.014	0.575	0.165	77.9%
India	0.096	0.827	0.533	0.217	67.0%
Kenya	0.037	0.819	0.618	0.073	87.3%
Malawi	0.021	1.107	0.507	0.038	93.6%
Average	0.212	0.979	0.705	0.307	63.8%
1/Average	4.720	1.021	1.418	3.260	69.2%

- The vast majority of income differences due to TFP differences.
- Capital-to-output ratios are relatively similar across countries.

Development accounting VII



- TFP differences are important for all countries.
- TFP differences explain almost all the income differences for the poorest countries.

Development accounting IIX

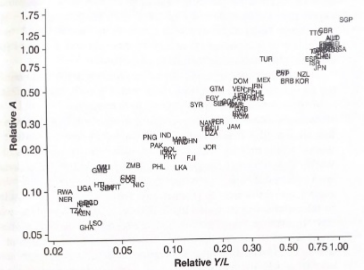
A different way to see the same point is to rewrite:

$$y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi u) \tag{18}$$

$$A(t) = \left(\frac{Y(t)}{K(t)}\right)^{\frac{\alpha}{1-\alpha}} \frac{y(t)}{\exp(\psi u)}, \tag{19}$$

i.e., ask what technology level do we require to explain the observed output per capita. We will express this again relative to the U.S.

Development accounting IX



As expected, there is a strong correlation between output per capita and the inferred TFP, i.e., other factors explain relatively little.

Success of the Solow model?

Taken together, the Solow model's explanation for cross-country differences in income per person are unsatisfactory:

- The endogenous component, $\frac{K(t)}{Y(t)}$, explains almost no variation.
- The exogenous component, $A(t)$, explains the vast majority of differences.

Steady state growth

Steady state growth over time

- So far, we have looked at cross-country differences in output per capita.
- We are now going to look at what explains growth within a country over time that is in steady state.
- We have already seen that the model is consistent with the Kaldor facts.
- Here, we will look at further implications of the model.

Decomposing output over time

We follow the framework proposed by Solow (1957). Similar to development accounting, we start again with the aggregate production function:

$$Y(t) = K(t)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \tag{20}$$

$$H(t) = L(t) \exp(\psi\, u(t)). \tag{21}$$

Now take logs and take the derivative with respect to time:

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha \frac{\dot{K}(t)}{K(t)} + (1-\alpha)\left[\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} + \psi \frac{\partial u(t)}{\partial t}\right]. \tag{22}$$

Output per capita over time

Instead of total output, we can also look at output per capita:

$$y(t) = \frac{Y(t)}{L(t)} = \left(\frac{K(t)}{L(t)}\right)^{\alpha} (A(t)\exp(\psi\, u(t)))^{1-\alpha} \tag{23}$$

$$\frac{\dot{y}(t)}{y(t)} = \alpha\frac{\dot{k}(t)}{k(t)} + (1-\alpha)\left[\frac{\dot{A}(t)}{A(t)} + \psi\frac{\partial u(t)}{\partial t}\right]. \tag{24}$$

The intuition is very simple. Output per worker grows either because capital per worker is growing (capital deepening), the quality of the workforce is growing, or technology is growing.

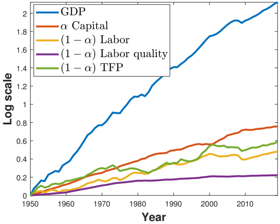
Quantifying the contribution of education

$$\frac{\dot{y}(t)}{y(t)} = \alpha \frac{\dot{k}(t)}{k(t)} + (1-\alpha)\left[\frac{\dot{A}(t)}{A(t)} + \psi \frac{\partial u(t)}{\partial t}\right]. \qquad (25)$$

Growth accounting takes a Neo-classical view to measure $\psi \frac{\partial u(t)}{\partial t}$

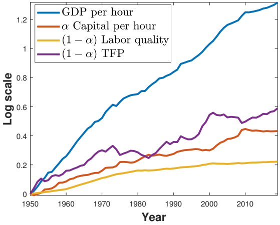
- Each worker is paid his/her marginal product.
- E.g., when a college worker earns twice as much as a high-school dropout, he/she is twice as productive.
- We can measure average wages (marginal products) by education groups for each year and compute changes in education shares and average wages over time.

Output growth in the U.S.



- Capital accumulation is the number one contribution for output growth in the U.S.
- A better educated workforce is relatively unimportant.
- The growth slowdown since 1970 is mostly due to low TFP growth. We observe some gains since 2010.

Output per hour growth in the U.S.



Also quantitatively, the model does a good job. As education is no longer constant, we should have $g_k > g$:

$$k(t)^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} A(t)\exp(\psi\, u(t)) \tag{26}$$

$$\left(\frac{\dot{k}(t)}{k(t)}\right)^* = \frac{\dot{A}(t)}{A(t)} + \psi\frac{\partial u(t)}{\partial t}. \tag{27}$$

This is the case (Log point changes 1.07 vs. 0.99).

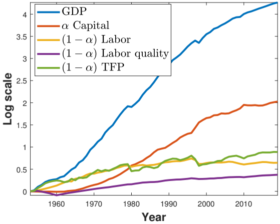
Convergence to steady state

Convergence to steady state

The Solow model also makes predictions about output growth in countries that converge to their new steady state:

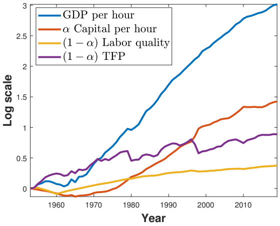
- Countries growing fast should do so temporarily because of rapid capital accumulation.
- Countries growing negatively should do so temporarily because of rapid capital decumulation.
- Percentage changes in the capital-to-output ratio should be largest early in the transition.

A growth miracle: output growth in Korea



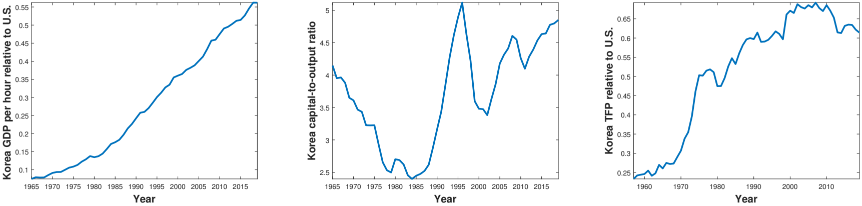
Korea grew much quicker than the U.S. (Log scale 4.3 vs. 2.1).

Output per hour growth in Korea



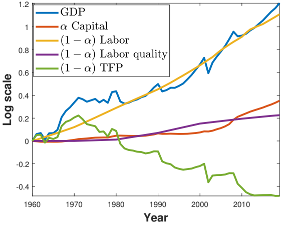
Again, consistent with the Solow model, we have $g_k > g$. In fact, we have $g_k >> g$ (Log scale 3.0 vs. 1.7). The result is even starker when taking 1965 as the starting point. Also consistent with the Solow model, g_y is falling over time.

Success of the Solow model?



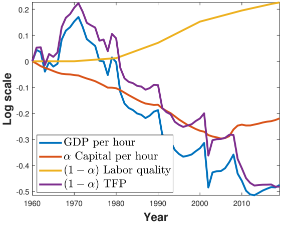
Again, having a theory of endogenous TFP is crucial to understand the data.

A growth disaster: output growth in Madagascar



- Madagascar was one of the poorest countries in 1960. Despite that, almost all output growth is due to labor growth.
- Capital growth is slow.
- TFP growth has been negative since 1970.

Output per hour growth in Madagascar



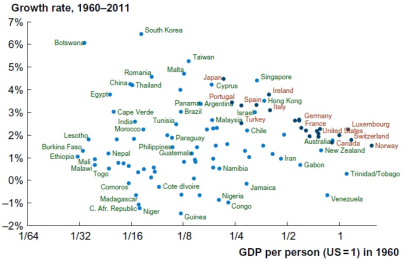
- A constant (positive) TFP growth rate is a poor assumption for Madagascar.
- In fact, *declining* TFP is key to understand declining output per hour.
- The capital-to-output ratio fell (!) from 1960–1970 and was constant since.

Do we observe convergence in living standards?

Does the world become more equal?

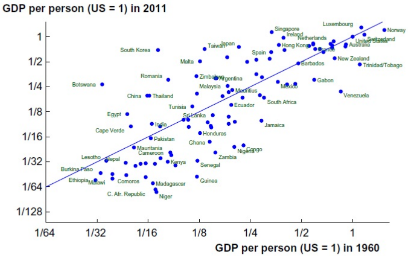
- We have seen that at any point in time, countries vary vastly in their income per capita.
- We may be interested in the question whether countries converge in their living standards over time.
- For convergence, we need that those countries who are relatively poor grow relatively quickly.

Convergence between countries



Looking at the world as a whole, we observe no general convergence between countries. It is not true that those countries which were poor in 1960 grew on average quicker than those countries who were rich in 1960. It is worth, however, to remember, that the picture would look different when looking at population weighted measures.

Convergence between countries II



As a result, those countries which were relatively rich in 1960 tend to be also relatively rich in 2011.

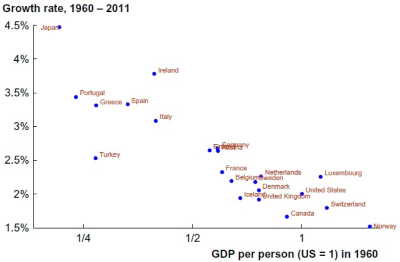
Is this inconsistent with the Solow model?

The world as a whole becoming more equal is referred to as *absolute convergence*.

It is important to recognize that the Solow model does not predict absolute convergence. Instead it predicts *conditional convergence*. Two countries with the same steady state should converge over time in GDP per capita.

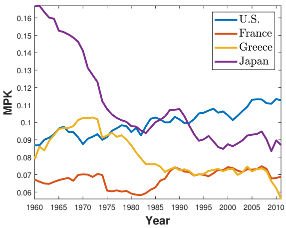
As we have seen, countries do not have all the same steady state. They differ in their savings rates, population growth rates, education, technology growth rates, and technology levels.

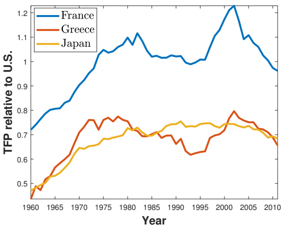
Is there conditional convergence?



The countries forming the OECD have relatively similar socio-economic structures and, hence, may be thought to have similar steady states. Baumol (1986) was the first to show that conditioning on this group of countries, we indeed observe convergence in living standards.

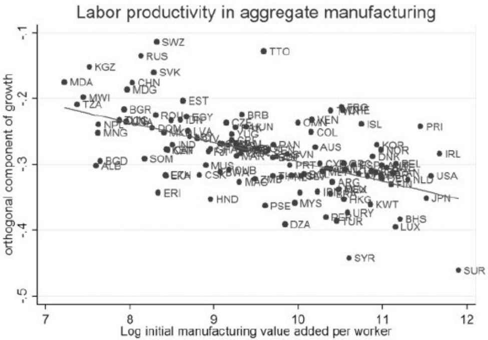
Is conditional convergence consistent with Solow model?





The Solow model predicts poor countries to have high MPK initially which converges over time. Only Japan is (partially) consistent with that prediction. Instead, productivity convergence is key.

Absolute convergence in manufacturing



Source: Rodrik (2013)

Manufacturing displays absolute convergence. One possible explanation is that the sector is globalized with multinational companies bringing their technology to different countries.

Final remarks: the good

- Differences in income per person correlate with population growth rates and savings rates differences.
- The model does a good job in explaining steady state growth.
- There is some evidence for conditional convergence.

Final remarks: the bad

In the end, we need a theory of TFP:

- The number one explanation for income differences across countries are TFP differences.
- Growth miracles are to a substantial part due to rapid TFP growth.
- Conditional convergence is linked to TFP convergence.

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