Endogenous growth: What matters are new ideas

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Growth Theory

Introduction

Why we need to understand the source of technological progress:

- Technological growth is the engine of growth in the Solow model.
- Cross-country differences in technology are key to understand output per worker differences.
- Growth miracles result to a large part from fast technological growth.

Introduction

- Romer (1990) presents a framework on how to analyze technological growth through research. For that work, he won the Nobel price.
- The framework is thought to model growth at the technological frontier through discovering new innovations.
- Afterward, we will consider how countries that lack behind may grow by copying existing ideas.

How ideas are discovered

A function for idea discovery

$$\dot{A}(t) = f\Big(\theta, L_A(t), A(t)\Big). \tag{1}$$

- θ is the research environment. This contains the quality of $R\&D$ institutions and the institutions that bring those ideas to the market.
- $L_A(t)$ is the number of researchers.
- The current stock of ideas, $A(t)$, may also affect how many new ideas are discovered.

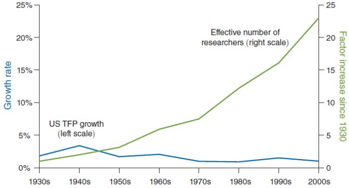
How the stock of ideas affects new discoveries

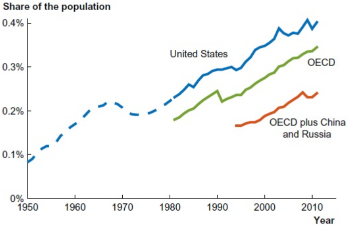
Two competing views:

- Standing on the shoulders of giants: As great innovators before us have come up with great ideas, it becomes easier to develop new ideas. For example, it would have been impossible to travel to the moon, if Newton and Leibniz had not invented calculus.

- The easiest fruits are already picked: Imagine there are a lot of new ideas out there but discovering each contains a varying degree of difficulty. For example, in medicine, doctors washing their hands lead to a drastic decline in mortality, yet discovering new cancer treatments has proven difficult. Naturally, we will begin by discovering easy ideas first.

R&D input and output in the data

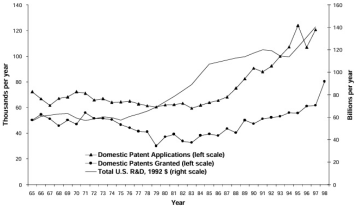




Source: Bloom et al. (2020) and Jones (2016)

- Over time, the rate of innovation, i.e., technological progress is close to constant.
- However, we have increased the number of inputs, i.e., the number of R&D workers.
- This suggests that the stock of existing ideas slows down the growth rate of new ideas.

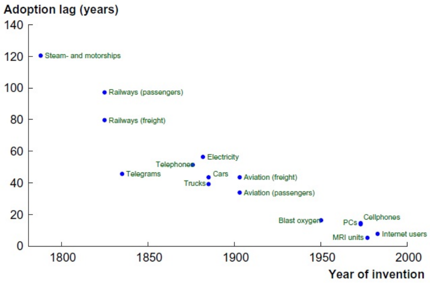
Falling R&D output productivity



Source: Jaffe (1999)

- Despite the number of researchers increasing in the U.S. exponentially, the number of new patents is close to flat with an increase in the 1980s.
- Even that did not increase productivity growth suggesting that innovations become more marginal.

Speed of adoption

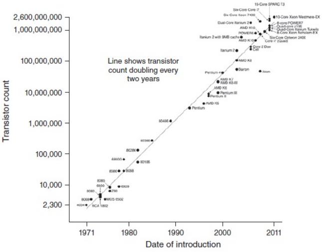


Source: Jones (2016)

- Once a new idea is developed, it still needs to be adopted by the economy.
- The data suggests that today, we are quicker in adopting new ideas than in the past.

Micro-level evidence

- Bloom et al. (2020) look at three cases where they can observe the input and output in the ideas production function.
- This allows them to study changes in research productivity within a field over time.
- They find that research productivity is falling by around 8% per year.



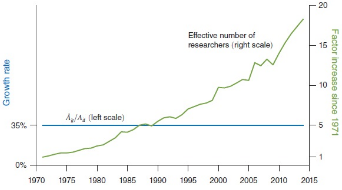
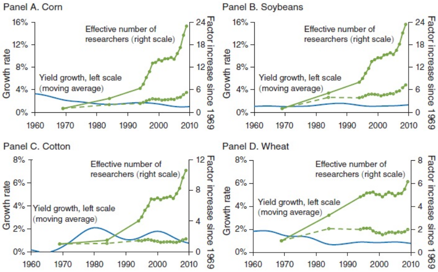


Figure 4. Data on Moore's Law

Source: Bloom et al. (2020)

- Since the 1970s, the number of transistors on a CPU doubles every 2 years implying a yearly research output of 35%.
- Using R&D and wage data from the major chip manufacturers, today we need 18 times more input for the same growth than in the 1970s.

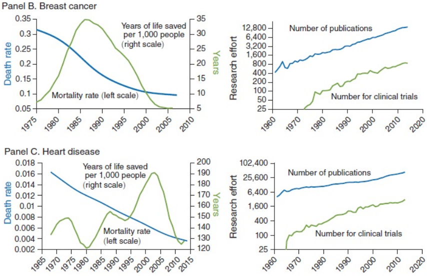
Crop yield



Source: Bloom et al. (2020)

- The increase in crop yields was relatively constant since the 1960s.
- At the same time, the number of researchers trying to increase crop yields has risen sharply.
- The data suggests a 6% yearly productivity decline.

Health



Source: Bloom et al. (2020)

- Improvements in health outcomes show no exponential growth.
- However, the number of researchers, again, has increased drastically.

The model

Do we need a new framework? Non-rivalrous good

It may be tempting to think about technology as just another input we can accumulate (like capital) and just modify our Solow model. This, however, misses a key aspect of technology: Different from capital, technology is a non-rivalrous good. If one person comes up with a better idea of producing, everyone could, in principle, use that idea:

- A newly discovered computer algorithm could be used by all producers/consumers.
- A newly discovered drug could benefit all sick people.
- Just-in-time delivery could be employed by all firms.
- A more efficient airplane design could be used by all firms.

Do we need a new framework? Increasing returns to scale

Non-rivalry often leads to increasing returns to scale:

- Producing the first unit is expensive as a lot of R&D needs to be invested.
- Once the new idea is discovered, producing more units comes at relatively cheap (sometimes zero) marginal costs. For example:

$$f(x) = 100(x - F). \tag{2}$$

- Hence, $f(ax) > af(x)$, i.e., we have increasing returns to scale.

Do we need a new framework? Imperfect competition

With increasing returns to scale, markets usually give rise to imperfect competition:

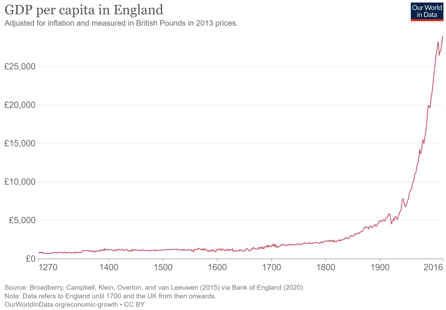
- The average costs are always higher than marginal costs. Hence, $p = MC$ would result in negative profits and no firms entering the market.
- Instead, imperfectly competitive firms need to make a profit on each unit of production to recuperate the large fixed costs.
- This requires that the good is excludable. Otherwise, the government will have to do the innovations, e.g., innovations in national defense.

How can firms make profits with their innovations?

- Some new ideas may be hard to copy. For example, the Coca Cola recipe was a trade-secret for a long time. Also firm organization innovations, such as new management styles may be hard to observe and difficult to copy from one firm to another.

- In other cases, e.g., new drugs, reverse engineering and copying may be relatively simple. To give firms and individuals nevertheless an incentive to innovate, the government grants temporary monopolies to innovations:
  - Patents for tangible ideas.
  - Copyrights for non-tangible ideas.

Can patenting explain the growth take-off?



- Remember that we did not observe growth in output per worker since relatively recently.
- England established a patent law in 1624 but the diffusion was slow.
- Some economists believe that spreading intellectual property protection was a major contributor to the industrial revolution.

The Romer model

Romer (1990) proposes a model where new ideas add to the existing stock of ideas. Conceptually, it is simplest to have a model with four different sectors:

- The household sector saves and accumulates the aggregate, homogeneous capital stock.
- Researchers develop new product designs taking the form of capital goods, e.g., the design of a new computer.
- An intermediate firm buys this design and uses capital from the household sector to turn it into a productive capital good on which it holds a patent.
- Finally, a large number of final goods producers buy these capital goods and produce the final good under perfect competition.

Final good production

To produce the final output good, we require differentiated capital goods. For example, an economy needs:

- assembly lines
- robotic arms
- stamping machines
- welding tools
- car chassis
- combustion engines

Final goods production II

The final goods producer use production labor and a measure of $A(t)$ available capital goods to produce the final output good:

$$Y(t) = L_Y(t)^{1-\alpha} \int_0^{A(t)} x_j(t)^{\alpha} dj. \tag{3}$$

Note, the function has constant returns to scale with respect to capital and labor. For a given amount of capital goods, A, doubling L_Y and each x_j will double output.

Optimal input demand

Given the wage, w, and the prices of each capital good, p_j, the final goods producer choose the quantity of each capital good and labor to maximize profits (the price of the final output good is 1, and omitting the time indexes):

$$\max_{L_Y, x_j} \left\{ \Pi = L_Y^{1-\alpha} \int_0^A x_j^\alpha dj - wL_Y - \int_0^A p_j x_j dj \right\}. \tag{4}$$

$$\frac{\partial \Pi}{\partial L_Y} = (1-\alpha) L_Y^{-\alpha} \int_0^A x_j^\alpha dj - w = 0 \tag{5}$$

$$w = (1-\alpha)\frac{Y}{L_Y}. \tag{6}$$

$$\frac{\partial \Pi}{\partial x_j} = \alpha L_Y^{1-\alpha} x_j^{\alpha-1} - p_j = 0 \tag{7}$$

$$p_j = \alpha L_Y^{1-\alpha} x_j^{\alpha-1}. \tag{8}$$

Capital good producers

The capital good producers buy patent designs from researchers at a fixed price P_A. Once they have the design, they produce capital goods by transforming capital from the household sector into productive capital goods at a one-to-one rate. The price of capital is r:

$$\max_{x_j} \{\pi_j = p_j(x_j)x_j - rx_j\}, \tag{9}$$

where $p_j(x_j)$ is given by the demand function (8).

Optimal price

$$\frac{\partial \pi_j}{\partial x_j} = \frac{\partial p_j(x_j)}{\partial x_j} x_j + p_j - r = 0 \tag{10}$$

$$0 = \frac{\frac{\partial p_j(x_j)}{\partial x_j} x_j}{p_j} + 1 - \frac{r}{p_j}. \tag{11}$$

From (8) we have

$$\frac{\frac{\partial p_j}{\partial x_j} x_j}{p_j} = \frac{(\alpha - 1)\alpha L_Y^{1-\alpha} x_j^{\alpha-2} x_j}{\alpha L_Y^{1-\alpha} x_j^{\alpha-1}} \tag{12}$$

$$= \alpha - 1. \tag{13}$$

Optimal price II

Putting things together:

$$p_j = \frac{1}{\alpha} r > r. \tag{14}$$

Imperfect competition implies that the capital goods producers make a profit on each unit they sell:

$$\pi_j = \frac{1}{\alpha} r x_j - r x_j = x_j r \frac{1-\alpha}{\alpha}. \tag{15}$$

The value of a patent

The implied costs for having a patent for one period is its price times the one-period return of an alternative investment: rP_A. Its one-period benefit is the flow profit plus the change in the value of the patent over that period:

$$rP_A = \pi + \dot{P}_A. \tag{16}$$

One can show that this solves for

$$P_A = \frac{\pi}{r-n}. \tag{17}$$

The research sector

The number of ideas that a single researcher discovers in a period depends on

- his productivity, θ, i.e., the research environment in an economy.
- *Standing on the shoulders of giants* suggests that the current stock of ideas has a positive effect: $A(t)^{\phi}$ with $\phi > 0$.
- The total number of researchers. If $\lambda < 1$, there is a stepping on your toes effect, otherwise, there are network effects.

$$\bar{\theta} = \theta A(t)^{\phi} L_A(t)^{\lambda - 1}. \tag{18}$$

The research sector II

To get to the total change in the number of new ideas in an economy, we have to multiply with the number of researchers, L_A:

$$\dot{A}(t) = \theta L_A(t)^{\lambda} A(t)^{\phi}. \tag{19}$$

The number of researchers

To an individual researcher, the value of being a researcher is $\bar{\theta} P_A$. This needs to equal her forgone wages from being a production worker:

$$\bar{\theta} P_A = (1-\alpha)\frac{Y}{L_Y}. \tag{20}$$

In steady state, one can show that this solves for

$$s_R = \frac{1}{1+\frac{r-n}{\alpha g^*}}. \tag{21}$$

- More efficient technological growth will encourage more researchers.
- A higher savings rate will decrease r and increase research.
- A higher population growth creates more demand for goods.

Market clearing

As each capital good x_j has the same cost and the same benefit, the equilibrium outcome is that all are produced in the same quantity:

$$x_j = x. \tag{22}$$

Let $K(t)$ be the total supply of capital in the economy. Then capital goods market clearing implies:

$$\int_0^A x_j dj = Ax = K \tag{23}$$

$$x = \frac{K}{A}. \tag{24}$$

The aggregate production function

Plugging the result into our final goods production function gives:

$$Y(t) = L_Y(t)^{1-\alpha} A(t) x(t)^{\alpha} \tag{25}$$

$$Y(t) = L_Y(t)^{1-\alpha} A(t) \left(\frac{K(t)}{A(t)} \right)^{\alpha} \tag{26}$$

$$Y(t) = (A(t) L_Y(t))^{1-\alpha} K(t)^{\alpha} \tag{27}$$

which is our familiar production function. The reason why more capital goods make the economy more productive is that it allows us to escape diminishing marginal returns. As a result, the function has increasing returns to scale with regard to all *three* production factors. The resulting real rental price of capital is

$$r(t) = \frac{\partial Y(t)}{\partial K(t)} = \alpha \left(A(t) L_Y(t) \right)^{1-\alpha} K(t)^{\alpha - 1} \tag{28}$$

$$= \alpha \frac{Y(t)}{K(t)}. \tag{29}$$

Household sector's labor supply

The household works either in the goods producing sector, L_Y, or the research sector, L_A:

$$L(t) = L_Y(t) + L_A(t) \tag{30}$$

$$L_A(t) = s_R L(t) \tag{31}$$

$$\dot{L}(t) = nL(t). \tag{32}$$

Household sector's income

The household is the owner of the capital stock and obtains income, $\tilde{Y}(t)$, from renting out the capital stock, working in the research sector, or working in the final good production sector:

$$\tilde{Y}(t) = r(t)K(t) + \int \pi_j(t)dj + L_Y(t)w(t), \tag{33}$$

where I have used the fact that income from being a researcher must equal total profits from capital producers. Substituting in profits and the wage yields

$$\tilde{Y}(t) = r(t)K(t) + \int x_j(t)r(t)\frac{1-\alpha}{\alpha}dj + (1-\alpha)Y(t) \tag{34}$$

$$\tilde{Y}(t) = r(t)K(t) + K(t)r(t)\frac{1-\alpha}{\alpha} + (1-\alpha)Y(t) \tag{35}$$

$$\tilde{Y}(t) = \frac{r(t)}{\alpha}K(t) + (1-\alpha)Y(t) \tag{36}$$

$$\tilde{Y}(t) = Y(t), \tag{37}$$

All output again flowing to the household in form of income simplifies our analysis. The household sector, as before, has a constant savings rate out of income:

$$\dot{K}(t) = sY(t) - \delta K(t), \tag{38}$$

which is our familiar law of motion.

The steady state

We begin our analysis with the steady state using the capital-to-output ratio:

$$z(t) = \frac{K(t)}{Y(t)} = \frac{K(t)}{K(t)^{\alpha}\left(A(t)L_Y(t)\right)^{1-\alpha}} \tag{39}$$

$$z(t) = \left(\frac{K(t)}{A(t)L_Y(t)}\right)^{1-\alpha}. \tag{40}$$

Implying a growth rate of

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)\left(n + g(t)\right). \tag{41}$$

Note, the key difference to the Solow model is that $g(t)$ is no longer constant, and we have now a theory for it!

The steady state II

As always, our second steady state condition comes from the capital accumulation equation:

$$\dot{K}(t) = sK(t)^{\alpha}\left(A(t)L_Y(t)\right)^{1-\alpha} - \delta K(t) \tag{42}$$

$$\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta \tag{43}$$

Solving for the steady state

Combining the equations and imposing a steady state

$$n + g(t) = \frac{s}{z^*} - \delta. \tag{44}$$

$$z^* = \left(\frac{K(t)}{Y(t)}\right)^* = \frac{s}{n + g(t) + \delta}, \tag{45}$$

which is a constant if $g(t)$ has a steady state to which we turn next.

The growth rate of new ideas

Recall the idea accumulation equation:

$$\dot{A}(t) = \theta L_A(t)^\lambda A(t)^\phi = \theta(s_R L(t))^\lambda A(t)^\phi. \tag{46}$$

When the current stock of ideas does not affect new idea creation, $\phi = 1$, we have

$$\frac{\dot{A}(t)}{A(t)} = \theta(s_R L(t))^\lambda, \tag{47}$$

i.e., for a constant $L(t)$, we always have constant exponential growth.

What if labor is not constant?

However, the data heavily suggests that $\phi < 1$: $\frac{\dot{A}(t)}{A(t)}$ is constant but $s_R L(t)$ is increasing over time. Going back to the ideas accumulation equation

$$\frac{\dot{A}(t)}{A(t)} = \frac{\theta(s_R L(t))^{\lambda}}{A(t)^{1-\phi}}. \tag{48}$$

The growth rate is still a constant if the numerator and denominator grow at the same rate, i.e.,

$$(1-\phi)\frac{\dot{A}(t)}{A(t)} = \lambda\frac{\dot{L}(t)}{L(t)} = \lambda n \tag{49}$$

$$\frac{\dot{A}(t)}{A(t)} = \frac{\lambda n}{1-\phi} = g^*, \tag{50}$$

which is indeed constant.

Key insights about productivity growth in steady state

$$g^* = \frac{\lambda n}{1 - \phi}. \tag{51}$$

- Constant technological progress is only possible through population growth (or a growing share of people doing research). More people provide more ideas which increases output, i.e., very different from the Solow model.
- The share of researchers in the population does not affect the growth rate of technology.
- Network effects, λ, and stepping on the shoulders of giants, ϕ, create faster technological progress.

Output per capita in steady state

$$Y(t) = K(t)^{\alpha} \left(A(t)L_Y(t)\right)^{1-\alpha} \tag{52}$$

$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)L_Y(t) \tag{53}$$

$$\frac{Y(t)}{L(t)} = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}(1-s_R)} A(t) \tag{54}$$

$$\left(\frac{Y(t)}{L(t)}\right)^{*} = \left(\frac{s}{n+g^{*}+\delta}\right)^{\frac{\alpha}{1-\alpha}} (1-s_R)A(t) \tag{55}$$

The final step is to derive $A(t)$ for which, again, we have now a theory.

The level of productivity in the Romer model

One can show that

$$A(t) = \left[A(0)^{1-\phi} + \frac{(1-\phi)}{n\lambda} \theta (s_R L(0))^{\lambda} \left[\exp(\lambda n t) - 1 \right] \right]^{\frac{1}{1-\phi}} . \qquad (56)$$

- A higher initial level of productivity, $A(0)$ implies higher productivity today.
- A higher population growth rate and stronger network effects imply higher productivity today.
- More productive researchers, θ, a higher share of researchers, s_R, or a larger initial workforce, $L(0)$, increase productivity.

Output per worker and population growth

Putting things together, we have in steady state:

$$\left(\frac{Y(t)}{L(t)}\right)^* = \left(\frac{s}{n+g^*+\delta}\right)^{\frac{\alpha}{1-\alpha}} (1-s_R) \left[A(0)^{1-\phi} + \frac{(1-\phi)}{n\lambda}\theta(s_R L(0))^{\lambda}\left[\exp(\lambda n t) - 1\right]\right]^{\frac{1}{1-\phi}}. \tag{57}$$

The population growth rate has an ambivalent effect on output per worker:

- The Solow effect: A higher n decreases the capital-to-output ratio.
- The Romer effect: A higher n increases the growth rate (and resulting stock) of ideas.

Output per worker and the share of researchers

$$\left(\frac{Y(t)}{L(t)}\right)^{*} = \left(\frac{s}{n+g^{*}+\delta}\right)^{\frac{\alpha}{1-\alpha}} (1-s_R) \left[A(0)^{1-\phi} + \frac{(1-\phi)}{n\lambda}\theta(s_R L(0))^{\lambda}\left[\exp(\lambda n t)-1\right]\right]^{\frac{1}{1-\phi}}. \tag{58}$$

The share of researchers has an ambivalent effect on output per worker:

- More researchers reduce the pool available to produce goods.
- More researchers lead to a higher stock of ideas and, hence, make those producing goods more productive.

Transition dynamics

$$\frac{\dot{A}(t)}{A(t)} = \frac{\theta(s_R L(t))^{\lambda}}{A(t)^{1-\phi}} \tag{59}$$

Our model suggests two ways to temporarily increase technological progress:

- Increase the share of researchers.
- Increase the efficiency of researchers.

Transition dynamics II

With $\phi < 1$, after a one-time increase in either of the two, we have

$$(1-\phi)\frac{\dot{A}(t)}{A(t)} > \lambda n, \tag{60}$$

i.e., the denominator grows quicker than the numerator. As the stock of ideas accumulates, $\frac{\theta(s_R L(t))^\lambda}{A(t)^{1-\phi}}$ falls and so does the rate of technological progress.

Solving for transition dynamics in productivity

We can use the solution for the level of productivity to solve for the transition dynamics:

$$\frac{\dot{A}(t)}{A(t)} = \frac{\theta(s_R L(t))^\lambda}{A(t)^{1-\phi}} \tag{61}$$

$$A(t) = \left[A(0)^{1-\phi} + \frac{1-\phi}{n\lambda}\theta(s_R L(0))^\lambda \left[\exp(\lambda n t) - 1\right]\right]^{\frac{1}{1-\phi}} \tag{62}$$

Combining the equations

$$\frac{\dot{A}(t)}{A(t)} = \frac{\theta(s_R L(t))^\lambda}{A(0)^{1-\phi} + \frac{1-\phi}{n\lambda}\theta(s_R L(0))^\lambda \left[\exp(\lambda n t) - 1\right]} \tag{63}$$

$$= \frac{\lambda n}{1-\phi} \frac{\theta(s_R L(t))^\lambda}{\frac{\lambda n}{1-\phi}A(0)^{1-\phi} + \theta(s_R L(t))^\lambda - \theta(s_R L(0))^\lambda} \tag{64}$$

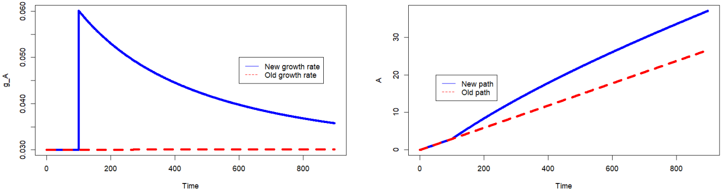
Solving for transition dynamics in productivity II

Rearranging yields

$$\frac{\dot{A}(t)}{A(t)} = \frac{\lambda n}{1-\phi} \frac{1}{\frac{\lambda n}{1-\phi} \frac{A(0)^{1-\phi}}{\theta (s_R L(t))^{\lambda}} + 1 - \left(\frac{L(0)}{L(t)}\right)^{\lambda}}. \tag{65}$$

- When starting in steady state: $\frac{A(0)^{1-\phi}}{\theta (s_R L(0))^{\lambda}} = \frac{1}{g^*} = \frac{1-\phi}{\lambda n}$, and $L(0) = L(t)$, then $\frac{\dot{A}(t)}{A(t)} = \frac{\lambda n}{1-\phi}$.
- For any $A(0), L(0)$, as time passes, $L(t)$ grows and $\frac{\dot{A}(t)}{A(t)} \mapsto \frac{\lambda n}{1-\phi}$.
- The lower $A(0)$ is relative to steady state, the faster is the initial growth rate.
- For any $A(0), L(0)$, increasing s_R, θ, ϕ, λ, or n increases $\frac{\dot{A}(t)}{A(t)}$.

Transition dynamics graphically



The temporarily higher growth rate in ideas leads to a permanently higher stock of ideas.

Transition dynamics for output per worker

As before, let us write output as a function of the capital to output ratio

$$y(t) = A(t)(1 - s_R)\left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} \tag{66}$$

$$\frac{\dot{y}(t)}{y(t)} = g(t) + \frac{\alpha}{1-\alpha}\frac{\dot{z}(t)}{z(t)} \tag{67}$$

with

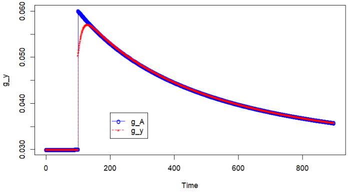
$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{s}{z(t)} - (1-\alpha)\left(n + g(t) + \delta\right). \tag{68}$$

Transition dynamics for output per worker II

$$\frac{\dot{y}(t)}{y(t)} = g(t) + \alpha \left[\frac{s}{z(t)} - (n + g(t) + \delta) \right] \tag{69}$$

Note, in steady state, $y(t)$ grows at rate g^*. A temporary increase in $g(t)$ increases the growth rate of $y(t)$ but it increases it initially by less than the increase in $g(t)$ because the capital to output ratio declines initially.

Transition dynamics graphically II



Schumpeterian growth

- In the Romer model, once discovered, designs are used forever.
- However, going back to Schumpeter, economists have thought about technological progress as better designs replacing older designs.
- For example,
  - touchscreen phones replaced keyboard phones.
  - the jet engine replaced propellers.
  - just-in-time delivery replaced large central storage units.
- Grossman and Helpman (1991) and Aghion and Howitt (1992) develop models of this creative destruction.

The model economy

As before, we will have the same four sectors:

- The household sector saves and accumulates the aggregate, homogeneous capital stock.
- Researchers develop new product designs taking the form of capital goods, e.g., the design of a new computer.
- An intermediate firm buys this design and uses capital from the household sector to turn it into a productive capital good on which it holds a patent.
- Finally, a final goods producer buys these capital goods and produces the final good under perfect competition.

Household sector

The household sector, as before, accumulates the aggregate capital stock and works in the final good sector or the research sector:

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{70}$$

$$L(t) = L_Y(t) + L_A(t) \tag{71}$$

$$L_A(t) = s_R L(t). \tag{72}$$

Final goods production

The final goods producer use production labor and the latest version of capital, x_i and productivity A_i to produce:

$$Y(t) = (L_Y(t)A_i(t))^{1-\alpha} x_i(t)^{\alpha}, \tag{73}$$

with x_i being newer and more productive than x_{i-1}. For example, x_i may be the jet engine and x_{i-1} the propeller, and A_i measures the productivity of the jet engine while A_{i-1} is the productivity of the propeller.

Optimal input demand

Given the wage, w, and the prices of each capital good, p_i, the final goods producer choose the quantity of each capital good and labor to maximize profits (the price of the final output good is 1, and omitting the time indexes):

$$\Pi = \max_{L_Y, x_i} \left\{ (L_Y A_i)^{1-\alpha} x_i^{\alpha} - wL_Y - p_i x_i \right\}. \tag{74}$$

$$w = (1-\alpha)\frac{Y}{L_Y} \tag{75}$$

$$p_i = \alpha (L_Y A_i)^{1-\alpha} x_i^{\alpha-1}. \tag{76}$$

Capital good producers

The capital good producers buy patent designs from researchers at a fixed price P_A. Once they have the design, they produce capital goods by transforming the capital from the household sector into productive capital goods at a one-to-one rate. The price of capital is r:

$$\max_{x_i} \left\{ \pi_i = p_i(x_i)x_i - rx_i \right\}, \tag{77}$$

which solves again for

$$p_i = \frac{1}{\alpha} r. \tag{78}$$

The research sector

Each new innovation is a constant improvement over the last innovation:

$$A_i(t) = (1+\gamma)A_{i-1}(t). \tag{79}$$

The probability that a researcher finds a new design is

$$\bar{\mu} = \frac{\theta L_A(t)^{\lambda-1}}{A_i(t)^{1-\phi}}. \tag{80}$$

Hence, the probability that a new design is discovered is

$$\bar{\mu} L_A(t) = \frac{\theta L_A(t)^{\lambda}}{A_i(t)^{1-\phi}}. \tag{81}$$

The value of a patent

When computing the value of a patent, we have to take into account that it loses all value with probability $\bar{\mu} L_A$:

$$rP_A = \pi + \dot{P}_A - \bar{\mu} L_A P_A. \tag{82}$$

One can show that this solves for

$$P_A = \frac{\pi}{r - n + \bar{\mu} L_A (1 - \gamma)}. \tag{83}$$

The more innovative a new patent is, γ, the more value it has. The more likely that a new innovation comes along, $\bar{\mu}$, the lower is the value.

The number of researchers

Using again a non-arbitrage condition, in steady state, one can show that the share of researchers is

$$s_R = \frac{1}{1 + \frac{r - n + \bar{\mu} L_A (1-\gamma)}{\alpha \bar{\mu} L_A}}. \tag{84}$$

A higher research efficiency makes me more likely to receive a patent and, thus, increasing the number of researchers but it also reduces its value.

Market clearing and aggregate output

As the final goods producer uses only the most productive capital good, we have that

$$x_i = K. \tag{85}$$

Substituting into its production function, we have again

$$Y(t) = (L_Y(t)A_i(t))^{1-\alpha} x_i(t)^{\alpha} \tag{86}$$

$$Y(t) = (A(t)L_Y(t))^{1-\alpha} K(t)^{\alpha} \tag{87}$$

Moreover, the interest rate is

$$r(t) = \alpha \frac{Y(t)}{K(t)}. \tag{88}$$

Income and capital accumulation

We have again for household income that

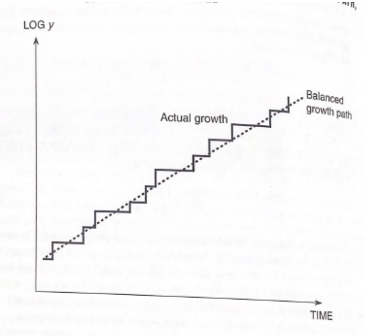
$$\tilde{Y}(t) = r(t)K(t) + \int \pi_j(t)dj + L_Y(t)w(t). \tag{89}$$

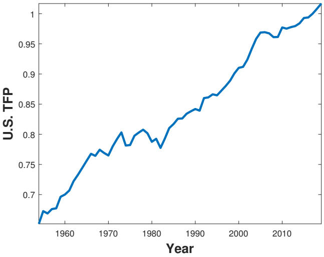
As prices and profits are unchanged, we have again $\tilde{Y}(t) = Y(t)$ and, hence

$$\dot{K}(t) = sY(t) - \delta K(t). \tag{90}$$

Hence, the aggregate economy behaves exactly the same as in the Romer model.

Steady state growth





One difference is that productivity is stochastic, it is only smooth in expectations.

$$\mathbb{E}A(t) = \bar{\mu}L_A(t)(1+\gamma)A(t-1) + (1-\bar{\mu}L_A(t))A(t-1) \quad (91)$$

$$\mathbb{E}A(t) = A(t-1)(1+\bar{\mu}L_A(t)\gamma). \quad (92)$$

Back to our three big questions

1. Why are we so rich and they so poor?
   - Different saving rates, population growth rates, and technologies for idea accumulation.
2. Why are there growth miracles?
   - Rapid accumulation of physical capital or ideas.
3. What are the engines of long run economic growth?
   - Population growth.

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